

# Fluctuations, correlations and the nonextensivity

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**Abstract:** Examples of joint probability distributions are studied in terms of Tsallis' nonextensive statistics both for correlated and uncorrelated variables, in particular it is explicitly shown how correlations in the system can make Tsallis entropy additive and that the effective nonextensivity parameter  $q_N$  decreases towards unity when the number of variables  $N$  increases. We demonstrate that Tsallis distribution of energies of particles in a system leads in natural way to the Negative Binomial multiplicity distribution in this system.

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## 1 Introduction

It is very well known fact that whenever in a single variable exponential distribution

$$f(x) = \frac{1}{\lambda} \cdot \exp\left(-\frac{x}{\lambda}\right) \quad (1)$$

parameter  $\lambda$  fluctuates according to gamma distribution, i.e.,  $\lambda \rightarrow \lambda' = \lambda'(\varepsilon) = \frac{\lambda}{\varepsilon}$ , where  $\varepsilon$  is distributed according to

$$g(\varepsilon) = \frac{1}{(q-1)\Gamma\left(\frac{1}{q-1}-1\right)} \left[\frac{\varepsilon}{q-1}\right]^{-2+\frac{1}{q-1}} \cdot \exp\left(-\frac{\varepsilon}{q-1}\right), \quad (2)$$

one obtains as result the following power-like distribution [1,2]:

$$h(x) = \int_0^\infty d\varepsilon g(\varepsilon) \left(\frac{\varepsilon}{\lambda}\right) \exp\left[-\frac{\varepsilon}{\lambda} \cdot x\right] = C_1 \left[1 - (1-q)\frac{x}{\lambda}\right]^{\frac{1}{1-q}}; \quad C_1 = \frac{2-q}{\lambda}, \quad (3)$$

called Tsallis distribution and characterized by parameter  $q$  [3,4] ( $q \in (1, 2)$ ), for  $q \rightarrow 1$  eq. (3) becomes the usual exponential distribution given by eq. (1)). Actually fluctuations can be also described by more general distributions (which induce wide spectrum of the so called *superstatistics* [1]), in all cases parameter  $q$  reflects the amount of fluctuations and is connected with their measure given by  $\omega = \text{Var}(\varepsilon)/\langle \varepsilon \rangle^2$ . In the case of eqs. (2,3) one has  $\langle \varepsilon \rangle = 2 - q$ ,  $\text{Var}(\varepsilon) = (q - 1)\langle \varepsilon \rangle$  and

$$\omega = \frac{\langle \varepsilon^2 \rangle}{\langle \varepsilon \rangle^2} - 1 = \frac{q-1}{2-q} \quad \text{or} \quad q = 1 + \frac{\omega}{1+\omega}. \quad (4)$$

This is result for the so called type B superstatistics [1]. In type A superstatistics, not accounting for  $\lambda$ -dependent normalization [2], one gets  $q = 1 + \omega$ . In this case large fluctuations were corresponding to large  $q$ , whereas eq. (4) describes all fluctuations using only limited range of parameter  $q$ , ranging from  $\omega = 0$  for  $q = 1$ , up to  $\omega \rightarrow \infty$  for  $q$  reaching its maximal allowed value of  $q \rightarrow 2$  (notice that for small values of  $\omega$  both approach lead practically to the same result:  $q \simeq 1 + \omega$ ).

Such distributions are widely used to characterize systems with stochastic processes and are often associated with the existence of long-range correlations, with memory effects and with nontrivial (multi)fractal phase space structure [3,4]. Especially important factor in getting such distributions are all possible intrinsic fluctuations which exist in the system under consideration [2] \*. In this paper we shall study examples of formal interrelations between fluctuations, correlations and nonextensivity, which can be of interest in physical applications, especially in the field of high energy multiparticle production processes (cf., [2,3,4] and references therein for other dynamical motivations). Notion of nonextensivity reflects the fact that Tsallis distribution (3) has also its origin in the so called nonextensive statistical mechanics (or information theory)

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\* In fact in recent work [5] it was argued that Tsallis distribution does not imply dynamics with correlated signals but rather with signals occurring in nonstationary intervals of time.

based on Tsallis entropy, which depends on parameter  $q$  (becoming for  $q \rightarrow 1$  the usual Boltzmann-Gibbs-Shannon entropy). It is normally nonextensive by amount proportional to  $q - 1$ , therefore  $q$  is named nonextensivity parameter [3,4]. Anticipating some applications we shall in what follows have in mind distributions of particles or their energies rather than some unspecified variables. Our goal is limited, the more general (but also very specialized) discussions on the role of correlations in obtaining Tsallis distributions can be found in [6], whereas in [7,8,9] one can find further discussions dealing with both fluctuations and correlations (in some specific scale-free approach) showing, among other things, that they can make Tsallis entropy additive. In next Section we show, using simple examples, that fluctuations in composite (but uncorrelated) systems can induce correlations and  $q \neq 1$ , i.e., nonextensivity. On the other hand fluctuations in correlated system can make it apparently uncorrelated and extensive (see also Appendix A). When applied to multiplicity distributions of particles produced in collision processes of all kinds it is shown that they convert the usual Poissonian distribution to the so called Negative Binomial ones found in all high energy processes [10] - this is shown in Section 3. The immediate practical applications of our results to some recent experimental data are presented in Appendix B.

## 2 Tsallis distributions: fluctuations *vs* correlations

### 2.1 Two random variable case

To introduce and discuss correlations one has to deal with at least two particles. Let  $x$  and  $y$  denote therefore two independent random variables, each following its own exponential distribution given by eq. (1) and let their joint probability distribution be given by  $f(x, y) = f(x) \cdot f(y)$ . The corresponding Tsallis distribution can be obtained either by fluctuating the parameter  $\lambda$  for each variable separately (in which case one obtains joint Tsallis distribution for *uncorrelated* random variables) or by fluctuating parameter  $\lambda$  jointly for both variables (in which case one gets Tsallis distribution for *correlated* random variables) [1]. It should be stressed that fluctuations lead always to Tsallis distribution, irrespectively of the presence or absence of correlations (their introduction does not change the statistics). We shall now discuss in more detail the uncorrelated

and correlated cases separately.

### 2.1.1 Uncorrelated random variables

In this case we fluctuate independently parameter  $\lambda$  in single particle probability distributions  $f(x)$  and  $f(y)$  as given by eq. (1), obtain in this way Tsallis distributions  $h(x)$  and  $h(y)$ , and finally the joint Tsallis probability distribution in the form of:

$$h(x, y) = h(x) \cdot h(y) = C^2 \cdot \left[ 1 - (1 - q) \frac{(x + y)}{\lambda} + (1 - q)^2 \frac{xy}{\lambda^2} \right]^{\frac{1}{1-q}}. \quad (5)$$

However, such distribution of *uncorrelated* variables can be also obtained starting from joint distribution of two *correlated* variables  $x$  and  $y$  and introducing to it suitable fluctuations in the way described before. Let us take, for example, the following two variable distribution,

$$f(x, y) = \left\{ \frac{\exp[\frac{1}{1-Q}]}{(Q-1)\Gamma\left(0, \frac{1}{Q-1}\right)} \right\} \cdot \exp\left[-\frac{x+y}{\lambda'} + (1-Q)\frac{xy}{\lambda\lambda'}\right], \quad (6)$$

where  $(Q - 1) \geq 0$  describes (negative) correlations between variables  $x$  and  $y$  present for  $Q \neq 1$ . It is characterized by the correlation coefficient  $\rho$ :

$$\rho = \frac{Cov(x, y)}{\sqrt{Var(x)}\sqrt{Var(y)}} = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\sqrt{Var(x)}\sqrt{Var(y)}} = \quad (7)$$

$$= \frac{(Q-1)\exp[\frac{1}{1-Q}] - E_1\left(\frac{1}{Q-1}\right)\left[1 + E_1\left(\frac{1}{Q-1}\right)\exp[\frac{1}{1-Q}]\right]}{(Q-1)\exp[\frac{1}{1-Q}] - QE_1\left(\frac{1}{Q-1}\right)} \quad (8)$$

(we have used here the exponential integral function  $E_1(z) = \int_1^\infty \frac{e^{-zt}}{t} dt$  and incomplete gamma function  $\Gamma(0, z) = \int_z^\infty \frac{e^{-t}}{t} dt$ ). Fig. 1 shows how  $\rho$  depends on  $Q$ . Let us now fluctuate in eq. (6) parameter  $\lambda'$  in the same way as in eq. (2) with  $q$  being measure of these fluctuations. It is straightforward to notice that one introduces in this way *positive* correlations between variables  $x$  and  $y$

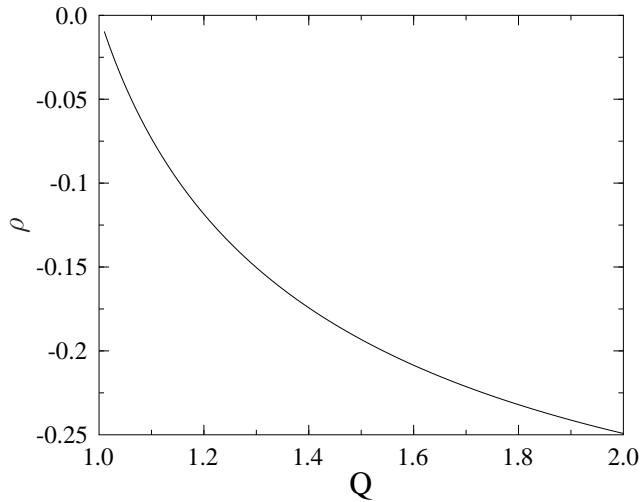


Fig. 1. Illustration of dependence of correlation parameter  $\rho$  as obtained by Eq, (8) on the nonextensivity parameter  $Q$ .

and that for some strength of these positive correlations given by the condition  $q = Q$  they cancel the negative correlations introduced in eq. (6) and we get uncorrelated variable distribution  $h(x, y)$  as given by eq. (5). This example shows that *effects of correlations and fluctuations can cancel each other in final result* \*\*. Let us notice at this point that correlations assumed in (6) which lead to (5) were recently widely used (cf. [11] and references therein) as example of the nonextensive rule for addition of energies resulting in Tsallis distribution. We argue that this is not true, namely that energies add always additively and formula used in [11] ( $\kappa(x, y) = x + y + axy$ ) leads to the formula for joint probability distribution, as our eqs. (5) or (6), which does not provide distribution of the sum of energies (cf. [12] and references therein for different mappings leading to different composition rules).

### 2.1.2 Correlated random variables

To get correlated random variables one fluctuates parameter  $\lambda$  in joint probability distribution of two variables,  $\{x, y\}$ , as given by  $f(x, y) = f(x) \cdot f(y)$ .

\*\*In Appendix A we provide another example of this kind when similar approach results also in extensivity of the final  $q$ -entropies of the original distributions. It should be stressed that here  $q \geq 1$  whereas in the example presented in Appendix A we have always  $q \leq 1$ .

Performing it in the same way as in (3) one gets the following joint Tsallis probability distribution:

$$\begin{aligned} h(x, y) &= \int_0^\infty d\varepsilon \left(\frac{\varepsilon}{\lambda}\right)^2 \exp\left[-\frac{(x+y)\varepsilon}{\lambda}\right] g(\varepsilon) = \\ &= C_2 \left[1 - (1-q)\frac{(x+y)}{\lambda}\right]^{\frac{q}{1-q}} \quad \text{with} \quad C_2 = \frac{2-q}{\lambda^2}. \end{aligned} \quad (9)$$

Notice that marginal probability distributions  $h(x) = \int h(x, y) dy$  and  $h(y) = \int h(x, y) dx$  have in this case also form of Tsallis distributions but with noticeably difference in the exponent, which lacks now the factor  $q$  present in (9), i.e., they are identical with eq. (3) where exponent is by unity higher than in (9) (being equal  $1/(1-q) = q/(1-q) + 1$ ). The corresponding mean value and variance of the marginal distributions are equal to

$$\langle x \rangle = \langle y \rangle = \frac{\lambda}{3-2q} \quad \text{and} \quad \text{Var}(x) = \text{Var}(y) = \frac{\lambda^2(2-q)}{(3-2q)^2(4-3q)}, \quad (10)$$

whereas the corresponding covariance and correlation coefficient corresponding to this joint probability distribution are equal to

$$\text{Cov}(x, y) = \frac{\lambda^2(q-1)}{(3-2q)^2(4-3q)} \quad \text{and} \quad \rho = \frac{q-1}{2-q} = \frac{1}{2-q} - 1. \quad (11)$$

It means then that correlation coefficient  $\rho$  is entirely given by the parameter  $q$ , which defines fluctuation of the random variable  $\varepsilon$  (actually, in this case it is just equal to the relative variance of  $\varepsilon$  given by  $\omega$ , cf. eq. (4)). Notice that we can now express the nonextensivity parameter  $q$  via the correlation coefficient:

$$q = 1 + \frac{\rho}{\rho + 1}. \quad (12)$$

In this way the correlation coefficient defines exponent in the Tsallis distribution. However, in any possible applications one has to remember that eq. (12) has been obtained only for correlations caused by fluctuations and therefore cannot be used for estimations of the role of correlations as such in Tsallis dis-

tributions. In Appendix A we present simple example of correlations making Tsallis entropy additive (cf. [7,8,9]).

## 2.2 *N* random variables case

It is straightforward to proceed to general case of  $N$  random variables  $\{x_{1,\dots,N}\}$  by fluctuating common parameter  $\lambda$  in the corresponding initial exponential distribution  $f(\{x_{1,\dots,N}\}) = \prod_i^N f(x_i)$  (with  $f(x_i)$  being given by eq. (1)) following prescription given by eq. (2). We get in this case following joint probability distribution:

$$\begin{aligned} h(\{x_{1,\dots,N}\}) &= \int_0^\infty d\varepsilon \left(\frac{\varepsilon}{\lambda}\right)^N \exp\left[-\frac{\varepsilon}{\lambda} \cdot \sum_{i=1}^N x_i\right] g(\varepsilon) = \\ &= C_N \left[1 - (1-q) \frac{\sum_{i=1}^N x_i}{\lambda}\right]^{\frac{1}{1-q}+1-N}, \end{aligned} \quad (13)$$

where

$$C_N = \frac{1}{\lambda^N} \prod_{i=1}^N [(i-2)q - (i-3)] = \frac{(q-1)^N}{\lambda^N} \cdot \frac{\Gamma\left(N + \frac{2-q}{q-1}\right)}{\Gamma\left(\frac{2-q}{q-1}\right)}. \quad (14)$$

It is straightforward to check that this distribution leads to marginal distributions in form of eq. (3) for each of variables considered.

Introducing  $N$ -particle nonextensivity parameter  $q_N$  one can formally rewrite eq. (13) as:

$$h(\{x_{1,\dots,N}\}) = C_N \left[1 - \frac{1-q_N}{1+(N-1)(1-q_N)} \cdot \frac{\sum_{i=1}^N x_i}{\lambda}\right]^{\frac{1}{1-q_N}} \quad (15)$$

where

$$\frac{1}{1-q_N} - \frac{1}{1-q_1} = 1 - N \quad \text{or} \quad q_N = 1 + \frac{q_1 - 1}{1 + (q_1 - 1)(N - 1)} \xrightarrow{N \rightarrow \infty} 1. \quad (16)$$

Notice that, irrespectively of how large are single variable fluctuations (represented by  $q_1$ ), they disappear in the multi-component systems with very large number of components  $N$  **\*\*\***.

### 3 Application to multiplicity distributions

#### 3.1 Boltzmann distribution and Poisson multiplicity distribution

Suppose that in some physical process one has  $N$  independently produced secondaries with energies  $\{E_{1,\dots,N}\}$ , each distributed according to Boltzmann distribution, i.e., according to eq. (1) with  $x = E_i$  and  $\lambda = \langle E_i \rangle$ . The corresponding joint probability distribution is then given by:

$$f(\{E_{1,\dots,N}\}) = \frac{1}{\lambda^N} \cdot \exp\left(-\frac{1}{\lambda} \sum_{i=1}^N E_i\right) = \prod_{i=1}^N \left[\frac{1}{\lambda} \cdot \exp\left(-\frac{E_i}{\lambda}\right)\right]. \quad (17)$$

For independent  $\{E_{i=1,\dots,N}\}$  the sum  $E = \sum_{i=1}^N E_i$  is then distributed according to gamma distribution,

$$g_N(E) = \frac{1}{\lambda(N-1)!} \cdot \left(\frac{E}{\lambda}\right)^{N-1} \cdot \exp\left(-\frac{E}{\lambda}\right) = g_{N-1}(E) \frac{E}{N-1}, \quad (18)$$

with distribuant equal to

$$G_N(E) = 1 - \sum_{i=1}^{N-1} \frac{1}{(i-1)!} \cdot \left(\frac{E}{\lambda}\right)^{i-1} \cdot \exp\left(-\frac{E}{\lambda}\right). \quad (19)$$

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**\*\*\*** In [13], in thermodynamical context and where  $q < 1$ , one has  $\frac{1}{q_N-1} - \frac{1}{q_1-1} = \frac{3}{2}(1-N)$  instead. Notice also that if  $(q_1-1)(N-1) \gg 1$  one has approximately that  $1/(q_N-1) = N \cdot 1/(q_1-1)$ , which seems to coincide with the notion of the *extensivity* of parameter  $\xi = 1/(q-1)$  discussed in [14,13]. Notice also that (15) is multivariable distribution rather then distribution of the sum of variables discussed recently in the context of the conjectured  $q$ -central limit theorem [15].

We look now for such  $N$  that  $\sum_{i=0}^N E_i \leq E \leq \sum_{i=0}^{N+1} E_i$ . Their distribution has known Poissonian form (notice that  $E/\lambda = \langle N \rangle$ ):

$$P(N) = G_{N+1}(E) - G_N(E) = \frac{\left(\frac{E}{\lambda}\right)^N}{N!} \cdot \exp(-\alpha E) = \frac{\langle N \rangle^N}{N!} \cdot \exp(-\langle N \rangle). \quad (20)$$

In other words, whenever we have variables  $E_1, E_2, \dots, E_N, E_{N+1}, \dots$  taken from the exponential distribution  $f(E_i)$  and whenever these variables satisfy the condition  $\sum_{i=0}^N E_i \leq E \leq \sum_{i=0}^{N+1} E_i$ , then the corresponding multiplicity  $N$  has Poissonian distribution (actually this is precisely the method of generating Poisson distribution in the numerical Monte-Carlo codes).

### 3.2 Tsallis distribution and Negative Binomial multiplicity distribution

Suppose now that in another process one has again  $N$  particles with energies  $\{E_{1,\dots,N}\}$  but this time distributed according to Tsallis distribution as given by eq.(13) (therefore, according to our previous discussion they cannot be independent but are correlated in some specific way),

$$h(\{E_{1,\dots,N}\}) = C_N \left[ 1 - (1-q) \frac{\sum_{i=1}^N E_i}{\lambda} \right]^{\frac{1}{1-q}+1-N}, \quad (21)$$

with normalization constant  $C_N$  given by eq. (14)). It means that, according to our reasoning behind eq. (13), there are some intrinsic (so far unspecified but summarily characterized by the parameter  $q$ ) fluctuations present in the system under consideration. Because variables  $\{E_{i=1,\dots,N}\}$  occur in the form of the sum,  $E = \sum_{i=1}^N E_i$ , one can perform sequentially integrations of the joint probability distribution (21) and, noting that

$$h_N(E) = h_{N-1}(E) \frac{E}{N-1} \quad \text{or} \quad h_N(E) = \frac{E^{N-1}}{(N-1)!} h(\{E_{1,\dots,N}\}), \quad (22)$$

arrive at formula corresponding to the previous eq. (18), namely

$$h_N(E) = \frac{E^{(N-1)}}{(N-1)!\lambda^N} \prod_{i=1}^N [(i-1)q - (i-3)] \left[ 1 - (1-q) \frac{E}{\lambda} \right]^{\frac{1}{1-q}+1-N} \quad (23)$$

with distribuant given by

$$H_N(E) = 1 - \sum_{j=1}^{N-1} \left\{ \frac{E^{j-1}}{(j-1)!\lambda^j} \prod_{i=1}^j [(i-1)q - (i-3)] \left[1 - (1-q)\frac{E}{\lambda}\right]^{\frac{1}{1-q}+1-j} \right\}. \quad (24)$$

As before, for energies  $E$  satisfying condition  $\sum_{i=0}^N E_i \leq E \leq \sum_{i=0}^{N+1} E_i$ , the corresponding multiplicity distribution is equal to

$$P(N) = H_{N+1}(E) - H_N(E) \quad (25)$$

and is given by the so called Negative Binomial distribution (NBD) (widely encountered in analyzes of high energy multiparticle production data of all kinds [10]):

$$\begin{aligned} P(N) &= \frac{(q-1)^N}{N!} \cdot \frac{q-1}{2-q} \cdot \frac{\Gamma\left(N+1+\frac{2-q}{q-1}\right)}{\Gamma\left(\frac{2-q}{q-1}\right)} \cdot \left(\frac{E}{\lambda}\right)^N \left[1 - (1-q)\frac{E}{\lambda}\right]^{-N+\frac{1}{1-q}} \\ &= \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \cdot \frac{\left(\frac{\langle N \rangle}{k}\right)^N}{\left(1 + \frac{\langle N \rangle}{k}\right)^{N+k}}, \end{aligned} \quad (26)$$

where the mean multiplicity and variance are, respectively,

$$\langle N \rangle = \frac{E}{\lambda}; \quad \text{Var}(N) = \frac{E}{\lambda} \left[1 - (1-q)\frac{E}{\lambda}\right] = \langle N \rangle + \langle N \rangle^2 \cdot (q-1). \quad (27)$$

(For different way of deriving of NBD by using Tsallis statistics see [16]). It is defined by the parameter  $k$  equal to:

$$k = \frac{1}{q-1}. \quad (28)$$

Notice that for  $q \rightarrow 1$  one has  $k \rightarrow \infty$  and  $P(N)$  becomes Poisson distribution whereas for  $q \rightarrow 2$  one has  $k \rightarrow 1$  and we are obtaining geometrical distribution. As we have notice before, fluctuations described by parameter  $q$  result also in specific correlations described by parameter  $\rho$  given by eq. (11). It means that

parameter  $k$  in NBD can be also expressed by the correlation coefficient  $\rho$  for the two-particle energy correlations (resulting from intrinsic fluctuations in the system), namely

$$k = \frac{\rho + 1}{\rho}. \quad (29)$$

It should be stressed at this point that result (28) coincides with our previous results in [17] where we have already obtained NBD from fluctuations of the mean multiplicity in the Poisson distribution. It means that such fluctuations are equivalent to fluctuations leading to eq. (13) which, following our reasoning presented in [2], we would like to attribute to the fluctuations of temperature  $T$  for the whole system **\*\*\*\***. We would like to close this Section with the remark that our result harmonizes with the known fact that whereas the generating function for Poisson distribution is exponential the corresponding one for the NBD has  $q$ -exponential form [18,16] (the relations between generating functions and probability distributions are the same as between  $P(E_i)$  and  $P(N)$  in our case, i.e.,  $P(E_i)$  plays the role of generating function for distribution  $P(N)$ ).

#### 4 Summary

To summarize, let us stress that fluctuations that can be described by gamma distribution (or equivalent to it in the sense discussed in [1] where term of *superstatistics* has been coined for this purpose) lead always to Tsallis distribution. On the other hand, not every fluctuation results in correlation. This is true only for fluctuations of the whole multicomponent system  $(x+y)$  or  $(\sum_i x_i)$ . Independent fluctuations of parameters  $\lambda_i$  in  $f(x, y) \sim [\exp(-x/\lambda_x)] \cdot [\exp(-y/\lambda_y)]$  lead to distribution  $h(x, y) = h(x)h(y)$  given by product of two Tsallis distributions with no correlations between variables  $x$  and  $y$ . Distributions obtained here differ from some many-particle distributions for composed systems  $\{x_{i=1, \dots, N}\}$  of the form  $\exp(\sum_i x_i) \longrightarrow [1 + (1 - q) \sum_i x_i]^{[1/(1-q)]}$ , which occur as apparently natural (and simple) generalization of the observation that in single component

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**\*\*\*\*** To make this point more transparent let us notice that because  $\langle N \rangle = E/\lambda$  therefore fluctuation of  $\langle N \rangle$  in Poisson distribution in [17] is equivalent (for fixed  $E$  as in our case) to fluctuation of  $1/\lambda$ , i.e., in our case to fluctuation of  $\langle E_i \rangle$ .

systems fluctuations lead to the replacement  $\exp(x) \longrightarrow [1 + (1 - q)x]^{[1/(1-q)]}$ , (and were also used to discuss correlations generated this way [11]) but they do not lead to correct (marginal) single particle distributions. Finally, we have proved that energy correlations introduced in multiparticle system by fluctuations (which can be traced to fluctuations of temperature in this system, which is the place where energy is converted into observed particles in process known as *hadronization*) result in changing the corresponding multiplicity distributions from Poisson to Negative Binomial ones and that this is equivalent to introducing fluctuations of the mean multiplicities in the Poisson distribution. The possible application of our findings to some recent data on multiparticle production processes is presented in Appendix B.

## A Example of extensive Tsallis entropy

In [2] we have shown that nonextensivity leading to Tsallis statistics [3] and characterized by parameter  $q$  can be caused by some intrinsic fluctuations existing in the physical system under consideration. The corresponding Tsallis entropy is nonextensive. However, as discussed in [4,7,8,9] (cf. also [19]) extensivity depends not only on the specific form of the entropy function used but also on the composition law according to which given composed system is formed out of its subsystem, i.e., on their possible correlations. In fact it is easy to demonstrate [4,7,8,9,19]) that by introducing to the physical system some specific correlations (for example, correlations that are strictly or asymptotically scale invariant [7,8,9,19]) one can make the corresponding entropy becoming extensive. In what follows we shall illustrate this point by using as example simple gaussian probability distributions for single and two correlated variables.

Let the single variable  $x$  distribution be of the gaussian form (with  $\sigma = \sqrt{Var(x)}$  being a parameter)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right). \quad (\text{A.1})$$

Let us correlate this variable with another variable,  $y$ , using two variable gaussian distribution with correlations provided by parameter  $\rho$  as defined in Eq.

(7):

$$f(x, y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2\sigma^2(1-\rho^2)}\right]. \quad (\text{A.2})$$

It is properly normalized, i.e.,  $\int \int f(x, y) dx dy = 1$ , and has properly defined marginal probabilities, namely  $\int f(x, y) dy = f(x)$ . It is obvious that  $x$  and  $y$  are not independent because  $f(x, y) \neq f(x)f(y)$ .

The corresponding Shannon entropies for  $f(x)$  and  $f(x, y)$  are

$$S_x = \frac{1}{2} [1 + \ln(2\pi) + 2 \ln(\sigma)], \quad (\text{A.3})$$

$$S_{x,y} = 1 + \ln(2\pi) + 2 \ln(\sigma) + \ln \sqrt{1 - \rho^2}, \quad (\text{A.4})$$

respectively, being nonextensive by amount

$$\delta S = S_{x,y} - 2S_x = \ln \sqrt{1 - \rho^2}, \quad (\text{A.5})$$

which depends on strength of correlation  $\rho$  (notice that  $\delta S \leq 0$ ) and becoming extensive only for uncorrelated system, i.e., when  $\rho = 0$ .

The corresponding Tsallis entropies are equal to:

$$T_x = \frac{1}{q-1} \left[ 1 - \frac{(2\pi)^{\frac{1-q}{2}} \sigma^{1-q}}{\sqrt{q}} \right] \quad (\text{A.6})$$

$$T_{x,y} = \frac{1}{q-1} \left[ 1 - \frac{(2\pi)^{1-q} (\sigma^2 \sqrt{1 - \rho^2})^{1-q}}{q} \right] \quad (\text{A.7})$$

$$\begin{aligned} \delta T &= T_{x,y} - 2T_x = \\ &= \frac{1}{q(1-q)} \left[ q + (2\pi)^{1-q} (\sigma^2 \sqrt{1 - \rho^2})^{1-q} - 2^{\frac{3-q}{2}} \pi^{\frac{1-q}{2}} \sigma^{1-q} \sqrt{q} \right]. \end{aligned} \quad (\text{A.8})$$

Notice that for uncorrelated variables, i.e., for  $\rho = 0$ , one gets in this case the usual result for Tsallis entropy:

$$\delta T = (1-q)T_x^2. \quad (\text{A.9})$$

However, it is obvious from eq. (A.8) (cf. also Fig. A.1) that one can always find such value of correlation  $\rho$  for which  $\delta T = 0$  and Tsallis entropy becomes extensive:

$$\sqrt{1 - \rho^2} = \left[ 2(2\pi)^{\frac{q-1}{2}} \sqrt{q} - (2\pi)^{q-1} q \right]^{\frac{1}{1-q}}, \quad (\text{A.10})$$

or, in crude approximation,

$$\sqrt{1 - \rho^2} \approx q; \quad 0 \leq q \leq 1. \quad (\text{A.11})$$

Notice that, according to eq. (A.5), both positive and negative correlations result in some (negative) imbalance of Shannon and Tsallis entropy. One has

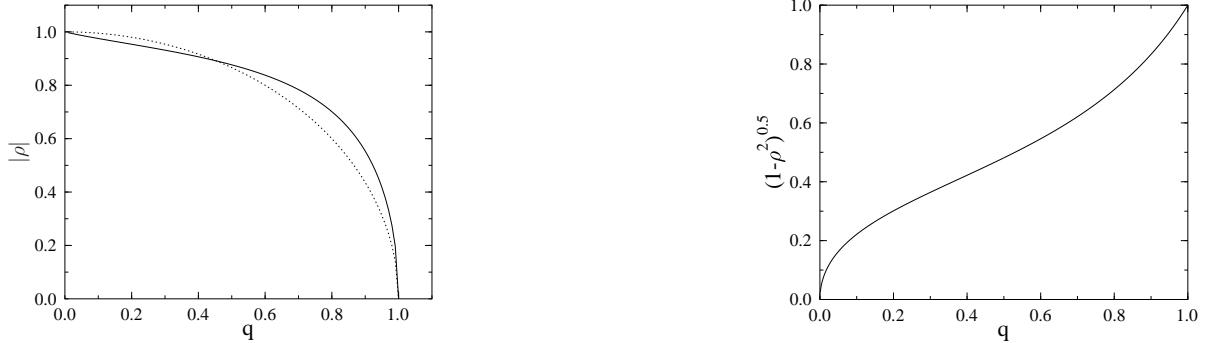


Fig. A.1. Left panel:  $|\rho|$  versus  $q$ : full line - exact formula (A.10); dotted line - approximate formula (A.11). Right panel:  $\sqrt{1 - \rho^2}$  versus  $q$ . Notice that  $0 \leq q \leq 1$ .

to stress at this point that what we have shown here is only a kind of formal exercise presented for illustration use only. It does not prove that correlations of some type lead to Tsallis type distributions. Variables can be correlated in any way and this does not depend on the distribution.

## B Example of possible practical applications

Results presented in this paper have practical application in the field of high energy multiparticle production reactions, especially those originated by collisions of heavy nuclei, which are of particular interest as potential source of

production of new state of matter, the so called *Quark Gluon Plasma* (QGP) (cf. references in [20,21,22,23]). Recently number of works [20,21,22,23] have demonstrated the existence in such reactions event-by-event fluctuations of the average transverse momenta  $\langle p \rangle$  per event. The following quantities were considered:  $Var(\langle p \rangle) / \langle \langle p \rangle \rangle^2$  and  $\langle \Delta p_i \Delta p_j \rangle / \langle \langle p \rangle \rangle^2$ . These quantities, as we advocate, are fully determined by  $\omega$  as defined by eq. (4), which in our case translates into fluctuations of the temperature  $T$  of hadronizing system - a vital observable when searching for QGP  $\star\star\star\star\star$ . The results obtained in [20,21,22,23] can be interpreted as fluctuations of the temperature  $T$  of the hadronic matter being produced (these are fluctuations for the whole event or its part (cluster) but not for the particular (single) particles in an event).

To show this let us consider the case of  $N_{ev}$  events with  $N_k$  particles in the  $k^{\text{th}}$  event. Introducing the following notation:

$$C_k = \sum_i^{N_k} \sum_j^{N_k} (p_i - \langle \langle p \rangle \rangle) \cdot (p_j - \langle \langle p \rangle \rangle), \quad (\text{B.1})$$

$$\langle \langle p \rangle \rangle = \frac{1}{N_{ev}} \sum_k^{N_{ev}} \langle p \rangle_k; \quad \text{where} \quad \langle p \rangle_k = \frac{1}{N_k} \sum_i^{N_k} p_i, \quad (\text{B.2})$$

we have that

$$C = \langle \Delta p_i \Delta p_j \rangle = \frac{1}{N_{ev}} \sum_k^{N_{ev}} \frac{C_k}{N_k(N_k - 1)}. \quad (\text{B.3})$$

Adding and subtracting the same term  $\langle p \rangle_k^2$  it can be written as

$$C = \frac{1}{N_{ev}} \sum_k^{N_{ev}} \left[ \frac{1}{N_k(N_k - 1)} \sum_i^{N_k} \sum_j^{N_k} p_i p_j + (\langle p \rangle_k^2 - \langle p \rangle_k^2) - \langle \langle p \rangle \rangle^2 \right]. \quad (\text{B.4})$$

---

$\star\star\star\star\star$ Generally speaking, analysis of transverse momenta  $p_T$  alone indicates very small fluctuations of  $T$ . On the other hand, as reported in [24], the measured fluctuations of multiplicities of produced secondaries are large (i.e., multiplicity distributions are substantially broader than Poissonian). Our analysis of NBD applied to observed multiplicity distributions show that this can result in large fluctuations of  $T$ , cf. [25].

If particles in the event are independent then

$$\frac{1}{N_k(N_k-1)} \sum_i^{N_k} \sum_j^{N_k} p_i p_j - \langle p \rangle_k^2 = 0 \quad (\text{B.5})$$

and we have that

$$C = \frac{1}{N_{ev}} \sum_k^{N_{ev}} (\langle p \rangle_k^2 - \langle \langle p \rangle \rangle^2) = \text{Var}(\langle p \rangle), \quad (\text{B.6})$$

or that

$$\frac{C}{\langle \langle p \rangle \rangle^2} = \frac{\text{Var}(\langle p \rangle)}{\langle \langle p \rangle \rangle^2} = \frac{\text{Var}(T)}{\langle T \rangle^2} = \omega. \quad (\text{B.7})$$

The above formulas can be checked against data obtained at Relativistic Heavy Ion Collider (RHIC) at Brookhaven Nat. Lab where  $Au$  nuclei are impinging at each other with center of mass energy 200 GeV per nucleon. Data taken by STAR experiment [22] for centrality  $30 - 40\%$  ( $N_{part} \sim 100$ ) give  $\omega = 4 \cdot 10^{-4}$ . The respective values for data obtained by other experiment, PHENIX, [23] are:  $2.2 \cdot 10^{-4}$ ,  $2.4 \cdot 10^{-4}$ ,  $3.6 \cdot 10^{-4}$  and  $4.9 \cdot 10^{-4}$  for the respective centralities:  $0 - 5\%$ ,  $0 - 10\%$ ,  $10 - 20\%$  and  $20 - 30\%$ . The more detailed analysis of the RHIC data along the line presented here is, however, out of the scope of the present paper and will be presented elsewhere.

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